Math Logic: Model Theory & Computability
Lecture 07

Definability. Let $\underline{A} := (A, \sigma)$ be a σ-structure.

Def. Let $P \subseteq A$ (think of elements of P as parameters).
For $n \in \mathbb{N}^+ := \{1,2,3,\dots\}$, we say that a set $B \subseteq A^n$ is $\emptyset$-definable if there is an extended σ-formula $\varphi(\vec{v})$, where $|\vec{v}| = n$, such that

$$B = \{\vec{a} \in A^n : \underline{A} \models \varphi(\vec{a})\}.$$

More generally, B is called P-definable if there is an extended σ-formula $\varphi(\vec{v}, \vec{u})$, where $|\vec{v}| = n$, and $\vec{p} \in P^m$, where $|\vec{u}| = m$, such that

$$B = \{\vec{a} \in A^n : \underline{A} \models \varphi(\vec{a}, \vec{p})\}.$$

A function $f : A^n \to A$ is called P-definable if its graph

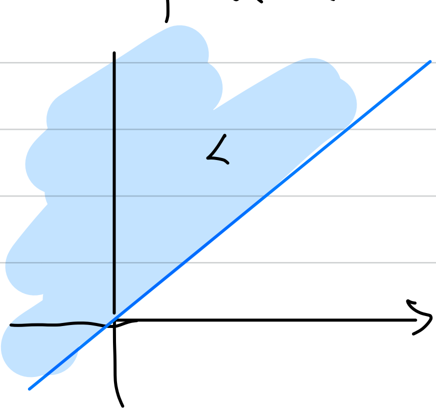
$$G_f := \{(\vec{a}, b) \in A^n \times A : f(\vec{a}) = b\}$$

is P-definable. Finally, a tuple $\vec{a} \in A^n$ is said to be P-definable if the singleton $\{\vec{a}\} \subseteq A^n$ is P-definable.
We say that a set/function/element is definable if it is A-definable.

Examples.

(a) In $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot)$, the set of positive reals is $\emptyset$-definable by the extended formula $\varphi_{>0}(x) := x \neq 0 \wedge \exists y\, (x = y \cdot y)$.
Also, the relation $< \subseteq \mathbb{R}^2$ is $\emptyset$-definable by the formula

$$\varphi_<(u,v) := \exists w\, (\varphi_{>0}(w) \wedge u + w = v).$$



Also the $r \mapsto -r$ function is $\emptyset$-definable by $\varphi_-(x,y) := (x + y = 0)$. Using this, we can redefine:
$\varphi_<(u,v) :=$ "$v - u > 0$" $:= \exists w\, (\varphi_-(u,w) \wedge \varphi_>(v+w))$.

(b) In any σ-structure $\underline{A} := (A, \sigma)$, every tuple $\vec{a} \in A^n$ is definable, in fact P-definable, where $P :=$ the elements of A that appear in $\vec{a}$. Indeed, if $\vec{a} := (a_1, \dots, a_n)$ and $P = \{a_1, \dots, a_n\}$, then taking $\vec{p} := \vec{a}$, we see that
$$\{\vec{a}\} = \{\vec{b} \in A^n : \underline{A} \models \varphi(\vec{b}, \vec{p})\}$$
where $\varphi(\vec{x}, \vec{y}) := \bigwedge_{i=1}^{n} x_i = y_i$, where $\vec{x} := (x_1, \dots, x_n)$, $\vec{y} := (y_1, \dots, y_n)$.

(c) If two sets are P-definable, then so is their union, also their complements. In particular, it follows from (b) that every finite subset $F \subseteq A^n$ is definable (but with parametres).

(d) In $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot)$, the set $\mathbb{N}$ is not definable, but we haven't developped the necessary tools (quantifier elimination) to prove this.

(e) In $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, every definable set is in fact $\emptyset$-definable. Indeed, each formula only uses finitely many parametres and each natural number $n \in \mathbb{N}$ is $\emptyset$-definable by the formula
$$\varphi(x) := \underbrace{S(S(\dots S}_{n}(0)) = n.$$

(f) Let $\mathrm{Th}(\underline{N}) := \{\varphi \in \mathrm{Sentences}(\sigma_{arithm}) : \underline{N} \models \varphi\}$. Each σ_{arithm}-word has a unique code $\in \mathbb{N}$, so let $\ulcorner \mathrm{Th}(\underline{N}) \urcorner \subseteq \mathbb{N}$ the set of corresponding codes. It's a theorem of Tarski that $\ulcorner \mathrm{Th}(\underline{N}) \urcorner$ is not definable in $\underline{N}$.

(g) Every function $\mathbb{N}^n \to \mathbb{N}$ that one can program on a computer is $\emptyset$-definable in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, in particular, $n \mapsto 2^n$. However, we need to postpone proving this until later.

(h) $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot)$. The set $\{\sqrt{2}, -\sqrt{2}\}$ is $\emptyset$-definable by the formu

In $\varphi(x) := (x^2 = 2)$, where $x^2 := x \cdot x$ and $2 := 1+1$. Thus, $\sqrt{2}$ is also $\emptyset$-definable by $\psi(x) := \varphi(x) \wedge \varphi_{>0}(x)$.

(i) However, in $\underline{\mathbb{C}} := (\mathbb{C}, 0, 1, +, \cdot)$, the set $\{\sqrt{2}, -\sqrt{2}\}$ is still $\emptyset$-definable by the same formula as above, but $\sqrt{2}$ is not $\emptyset$-definable, which again follows from the fact that $\underline{\mathbb{C}}$ admits quantifier elimination (a theorem we will prove later), hence the only $\emptyset$-definable sets are the sets of roots of polynomials, and their Boolean combinations, i.e. finite unions and complements.
Of course, $\sqrt{2}$ is $\{\sqrt{2}\}$-definable in $\underline{\mathbb{C}}$.

We already saw that advanced techniques like quantifier elimination is used to prove non-definability, but here is a cheap technique for this:

Prop. Let $\underline{A} := (A, \sigma)$, $\underline{B} := (B, \sigma)$ be σ-structures and $h : \underline{A} \to \underline{B}$ be an isomorphism. Then for each extended σ-formula $\varphi(\vec{v})$ and $\vec{a} \in A^{|\vec{v}|}$,

$$\underline{A} \models \varphi(\vec{a}) \text{ if and only if } \underline{B} \models \varphi(h(\vec{a})).$$

Proof. We prove by induction on the construction of φ. Let $\vec{a} \in A^{|\vec{v}|}$.

<u>Case 1: $\varphi := t_1 = t_2$.</u> Then we already know from lecture 5 that $h(t_i^{\underline{A}}(\vec{a})) = t_i^{\underline{B}}(h(\vec{a}))$, so $t_1^{\underline{A}}(\vec{a}) = t_2^{\underline{A}}(\vec{a})$ if and only if $t_1^{\underline{B}}(h(\vec{a})) = t_2^{\underline{B}}(h(\vec{a}))$, where $\Leftarrow$ holds because h is injective.

<u>Case 2: $\varphi := R(t_1, \dots, t_k)$</u> where $R \in \mathrm{Rel}_k(\sigma)$, $t_1, \dots, t_k$ are σ-terms. Then by the
$\underline{A} \models R(t_1(\vec{a}), \dots, t_k(\vec{a}))$ if and only if $\underline{B} \models R(t_1(h(\vec{a})), \dots, t_k(h(\vec{a})))$
by the definition of isomorphism.

Case 3: $\varphi := \varphi_1 \vee \varphi_2$. Follows easily by from the induction hypothesis.

Case 4: $\varphi := \neg \psi$. Follows easily by from the induction hypothesis.

Case 5: $\varphi := \exists u \psi$. Then $\psi(\vec{v}, u)$ is an extended formula, so by the induction hypothesis for any $a' \in A$,

$$\underline{A} \models \psi(\vec{a}, a') \quad \text{if and only if} \quad \underline{B} \models \psi(h(\vec{a}), h(a')).$$

Thus, $\underline{A} \models \exists u \, \psi(\vec{a}, u)$ if and only if there is $a' \in A$ s.t. $\underline{A} \models \psi(\vec{a}, a')$
if and only if there is $a' \in A$ s.t. $\underline{B} \models \psi(h(\vec{a}), h(a'))$
[here $\Uparrow$ is by surjectivity of h] if and only if there is $b \in B$ s.t. $\underline{B} \models \psi(h(\vec{a}), b)$
if and only if $\underline{B} \models \exists u \, \psi(h(\vec{a}), u)$. □

Cor. Let $\underline{A} := (A, \sigma)$ be a σ-structure and $D \subseteq A^n$ be a P-definable set. Then every automorphism $h : \underline{A} \to \underline{A}$ that fixes P pointwise (i.e. $h(p) = p$ for every $p \in P$) must fix D setwise (i.e. $h(D) = D$).

Proof. Let $\varphi(\vec{v}, \vec{u})$ be an extended σ-formula and $\vec{p} \in P^{|\vec{u}|}$ s.t.

$$D = \{\vec{a} \in A^n : \underline{A} \models \varphi(\vec{a}, \vec{p})\}.$$

Let $h : \underline{A} \to \underline{A}$ be an automorphism fixing P pointwise. Then

$$h(D) = \{h(\vec{a}) : \underline{A} \models \varphi(\vec{a}, \vec{p})\} \overset{\text{by the prop. above}}{=} \{h(\vec{a}) : \underline{A} \models \varphi(h(\vec{a}), h(\vec{p}))\}$$

$$(h \text{ fixes } \vec{p}) \quad = \{h(\vec{a}) : \underline{A} \models \varphi(h(\vec{a}), \vec{p})\} \overset{h \text{ is a bijection}}{=} \{\vec{b} \in A^n : \underline{A} \models \varphi(\vec{b}, \vec{p})\} = D.$$ □

Examples. (a) In $(\mathbb{Z}, 0, +)$, the set $\mathbb{N}$ is not $\emptyset$-definable because $h : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto -z$ is an automorphism which maps $\mathbb{N}$ to $-\mathbb{N}$, hence does not fix $\mathbb{N}$ setwise.

(b) In $\underline{Z} := (\mathbb{Z}, 0, +, \cdot)$, the set $\mathbb{N}$ is $\emptyset$-definable due to Lagrange's Four Squares theorem, which states that every natural number is a sum of four squares of integers, so

$$\mathbb{N} = \{ n \in \mathbb{Z} : \underline{Z} \models \exists x \exists y \exists z \exists v \, (x \cdot x + y \cdot y + z \cdot z + v \cdot v = n) \}.$$